FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

Algebra can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

This numerical question on the roots of a polynomial proved well understood. The most efficient approach was to write down the value of b from the given information and use the formula for the sum of squares of roots to find c. Numerical errors arose most often in the calculation of d. Those who substituted each root into the original equation and summed in order to use the given values were more likely to be correct than those who tried to use a formula for the sum of the cubes – this was sometimes inaccurately remembered.

Question 2

(a) This was often well done. Some candidates recognised that $n(n + 1)$ was a common factor and so had to simplify only a quadratic bracket. Those who multiplied out all brackets rarely achieved the final factorised form.

(b) The breaking down into three partial fractions was often done well. Three terms meant that the cancelling for the method of differences stretched over three lines and the pattern needed to be shown. There were some elegant solutions where the cancelling was clear from the layout. All candidates would find it helpful to set out their work with enough space to explore the way the terms fit together. In particular, the last terms must be shown.

(c) This part was well done by most candidates.

Question 3

(a) There was evidence that structure of a proof by induction was well understood and the hypotheses were written correctly, with the algebraic statement shown. The logarithms involved in this example meant that candidates needed to be ready to use, for example, $\ln(a_k)^3 = 3\ln(a_k)$. For the base case (where n = 2) this meant recognising that ln 8 = 3 ln2. The progression from the k term to the k + 1 term was most efficiently done by using the given result that $\ln(a_{k+1}) = 3\ln(a_k + \frac{1}{a_k}) > 3\ln a_k$ before using the inductive hypothesis.

(b) There were few completely correct solutions. Candidates should take great care when working with inequalities, remembering that multiplication by a negative quantity changes the direction of the inequality and so subtraction is not valid.

Question 4

(a) The terms stretch and rotation were well known and most applied them in the correct order.

(b) The method for finding invariant lines is well understood and many candidates could distinguish clearly between object and image points. The zero discriminant for there to be only one invariant line was accurately applied. The resulting trigonometrical equation was usually solved to give at least one of the two solutions.

Question 5

As with all vector questions, mistakes in arithmetic and with signs lost many accuracy marks. Candidates should develop the habit of checking carefully.

(a) Those who recognised the form in which the plane was given, efficiently found the normal and used the point to find the equation in the required form. A few used simultaneous equations.

(b) In general this was well done by most candidates.

(c) The method for finding the angle between two vectors is clearly well understood. In this case the angle required was between the line and the plane, so the angle between normal and line had to be subtracted from the right angle. Candidates are advised to reread the question to check that they are giving the required angle.

(d) Most correct answers came from using the formula for the distance of a point from a plane, with the common errors being with signs.

Question 6

(a) Several candidates tried to find the coordinates of the point which is furthest from either the initial line or the half-line $\theta = \frac{\pi}{2}$. Those who found $\frac{dr}{d\theta}$ could form an appropriate trigonometric equation and solve it to find the relevant value of θ. Polar coordinates must be given in the form (r, θ).

(b) Many understood that the graph was a single loop in the first quadrant but only some recognised that the curve is tangential to $\theta = \frac{\pi}{2}$ at the pole.

(c) The correct formula with correct limits was applied here in most cases. The idea of using double angles to integrate squares of sin or cos was used accurately. Some candidates recognised that $\sin\theta\cos^2\theta$ can be integrated either by inspection or by using a substitution.

Question 7

(a) Most candidates could find the vertical asymptotes. The horizontal asymptote of $y = 0$ was included by the majority.

(b) This was well answered with only occasional errors in arithmetic.

(c) Almost all could find the intersections with the axes. There were many good graphs with a carefully drawn curve, with axes and asymptotes clearly labelled. Some candidates recognised the significance of the stationary points found in **part (b)** and drew a curve that approached the negative y axis from above.

(d) The concept of reflecting the graph in the x axis to find the graph of the modulus was well demonstrated and applied. The approach of the line to the asymptote always needs care.

The most successful way to deal with the inequality was to split it in two, and this was usually seen. It is often most straightforward to find the critical values of x by using equations rather than inequalities. There were 4 such values and exact answers were required, with the majority of candidates including the value $x = 0$. The set of values of x satisfying the given inequality can then be determined using the graph.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

Algebra can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

(a) Most candidates knew the term enlargement for the single transformation.

(b) The concept of multiplying area by the absolute value of the determinant was well understood. Candidates are advised to check which is the object triangle and which the image to avoid dividing by mistake.

(c) The usual method was to pre-multiply by the inverse of A, and errors were generally in arithmetic. A number of candidates used simultaneous equations.

(d) Some candidates chose a general point (x, y) and used the fact that it was invariant to set up simultaneous equations and show that the only solution was (0, 0). This approach was generally successful.

Question 2

The structure of a proof by induction is was demonstrated well by most candidates. The hypotheses were set up properly including the algebraic form and it was uncommon to see the wrong assumption that the result is true for all values of k. The differentiation was correct and the conclusion stated to make the implication clear.

Question 3

(a) Many candidates started by expressing $\left(\dfrac{r(r+2)}{(r+1)^2}\right)$ in partial fractions, but some did not include the logarithm. Some candidates used the laws of logs to write $\displaystyle\sum_{r=1}^{n}\ln\left(\frac{r(r+2)}{(r+1)^2}\right)=\sum_{r=1}^{n}(\ln r-2\ln(r+1))+\ln(r+2))$ which gave a structure to apply the method of differences. Three values of r need to be used to establish the pattern of cancellation.

A particularly neat solution was to write $\displaystyle\sum_{r=1}^{n}\ln\left(\frac{r(r+2)}{(r+1)^2}\right)=\sum_{r=1}^{n}(\ln r-\ln(r+1))-\sum_{r=1}^{n}(\ln(r+1)-\ln(r+2))$ and deal with the two summations separately.

Another approach was to use the properties of the logarithms to replace the summation with a product.

Whichever method is used, candidates are advised to set their work out with enough space so the pattern is clear. In particular, the final terms must be shown.

(b) Many realised that the sum to infinity is – ln2, although the most common result for S was 0. In this case it was common to change the direction of the inequality.

Question 4

(a) This part was well done by most candidates.

(b) Most candidates used the original equation to prove the result.

(c) The expansion of $(\alpha+r)^3$ was well done and candidates went on to use the results of **parts (a)** and **(b)** to find the sum $\displaystyle\sum_{r=1}^{n}1-6r-6r^2+3r^3$. The correct formulae were chosen from the list of formuale. Those candidates who recognised that the last three terms had the common factor $n(n+1)$ had to simplify a quadratic term to get to the given result. This was more successful than multiplying out and then trying to re-factorise.

Question 5

(a) The correct symmetrical shape was usually drawn.

(b) Most could efficiently set up and solve a quadratic in $\sin\theta$ to find at least one of the angles. Candidates are reminded that polar coordinates are expressed in the form (r,θ) Many could add the required line to the previous diagram, although some candidates did not recognise that the line was, in fact, $y = 2$.

(c) The best solutions involved use of the diagram on the question paper or a sketch to label and keep track of the areas being found. The correct formula was used but the limits of integration rarely corresponded to the region required. The actual integration was well done, with confident use of the double angle to deal with $\sin^2\theta$.

Question 6

(a) The asymptotes were usually correctly identified. Candidates using division to find an oblique asymptote are advised to continue the process until they have considered the constant term of the line.

(b) Whenever the answer is given in the question candidates need to give full detail in their solution to show that this is true. The most efficient method was to use the discriminant of a quadratic in x to find the values of y for which solutions are possible. It should be made clear in the working whether a negative discriminant (no solutions) or a non-negative discriminant (solutions possible) is being considered. Many factorised $y^2 - 12y$ or drew a sketch to show why they were giving the range $0 < y < 12$.

Several candidates attempted to find the stationary values of the curve. Showing that these are (0, 0) and (6, 12) does not show that there are no possible values of *y* between 0 and 12. This would need a check that they are maximum and minimum and detailed reference to the nature of the graph including the position of the asymptotes.

(c) There were many good graphs seen with a carefully drawn curve, and axes and asymptotes clearly labelled.

(d) (i) The concept of reflecting the graph in the *x* axis to find the graph of the modulus was well demonstrated and applied. The graph of $y = |x| - 3$ was usually correct. A few candidates omitted the part of this graph below the *x* axis and a number did not give all the points of intersection with the axes.

(ii) These values could be read from the graph. Candidates need to check that they are using the appropriate inequality.

Question 7

As with all vector questions, candidates should take care to avoid mistakes in arithmetic and loss of signs. Candidates should develop the habit of checking carefully.

(a) The question asked for the equation of the plane *OAB* but many found the equation of *ABC*. The method of using the cross product to find the normal and then substituting a point was accurately applied but candidates should check that they are giving the answer in the form required in the question.

(b) This part was well done by many candidates, the only repeated error seen was when candidates did not divide by the magnitude of the normal.

(c) Most candidates found the angle between two vectors. In a few cases the final answer was spoilt by unnecessary subtraction from 90°.

(d) This part of the question is best approached by using clearly defined steps through the necessary stages, annotating the method as the working progresses. Candidates will find it helpful to define the positions of any new points they introduce and to state what lines they are using to avoid becoming lost in their own working.

Many found the direction of the common perpendicular first. To find its equation candidates needed to work with general points on both of the lines *OC* and *AB*, which involved using two parameters. Some candidates attempted to fix a point on one of the lines. Writing down the direction vector of this common perpendicular led to many errors in signs. Most stated that this vector was perpendicular to both *OC* and *AB* and produced simultaneous equations. These could be solved to find the parameters and hence the points and the equation of the line. A few put the vector equal to the direction of the common perpendicular rather than a multiple of it – if this method is used another parameter must be introduced. Several solutions omitted *r* = and so did not give an equation as required.

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Candidates should ensure that any sketch graphs are fully labelled and carefully drawn to show significant points and behaviour at limits.

Algebra can often be simplified by the use of common factors.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations.

Comments on specific questions

Question 1

This numerical question on the roots of a polynomial proved well understood. The most efficient approach was to write down the value of b from the given information and use the formula for the sum of squares of roots to find c. Numerical errors arose most often in the calculation of d. Those who substituted each root into the original equation and summed in order to use the given values were more likely to be correct than those who tried to use a formula for the sum of the cubes – this was sometimes inaccurately remembered.

Question 2

(a) This was often well done. Some candidates recognised that $n(n + 1)$ was a common factor and so had to simplify only a quadratic bracket. Those who multiplied out all brackets rarely achieved the final factorised form.

(b) The breaking down into three partial fractions was often done well. Three terms meant that the cancelling for the method of differences stretched over three lines and the pattern needed to be shown. There were some elegant solutions where the cancelling was clear from the layout. All candidates would find it helpful to set out their work with enough space to explore the way the terms fit together. In particular, the last terms must be shown.

(c) This part was well done by most candidates.

Question 3

(a) There was evidence that structure of a proof by induction was well understood and the hypotheses were written correctly, with the algebraic statement shown. The logarithms involved in this example meant that candidates needed to be ready to use, for example, $\ln(a_k)^3 = 3\ln(a_k)$. For the base case (where $n = 2$) this meant recognising that ln 8 = 3 ln2. The progression from the k term to the $k + 1$

term was most efficiently done by using the given result that $\ln(a_{k+1}) = 3\ln(a_k + \frac{1}{a_k}) > 3\ln a_k$ before using the inductive hypothesis.

(b) There were few completely correct solutions. Candidates should take great care when working with inequalities, remembering that multiplication by a negative quantity changes the direction of the inequality and so subtraction is not valid.

Question 4

(a) The terms stretch and rotation were well known and most applied them in the correct order.

(b) The method for finding invariant lines is well understood and many candidates could distinguish clearly between object and image points. The zero discriminant for there to be only one invariant line was accurately applied. The resulting trigonometrical equation was usually solved to give at least one of the two solutions.

Question 5

As with all vector questions, mistakes in arithmetic and with signs lost many accuracy marks. Candidates should develop the habit of checking carefully.

(a) Those who recognised the form in which the plane was given, efficiently found the normal and used the point to find the equation in the required form. A few used simultaneous equations.

(b) In general this was well done by most candidates.

(c) The method for finding the angle between two vectors is clearly well understood. In this case the angle required was between the line and the plane, so the angle between normal and line had to be subtracted from the right angle. Candidates are advised to reread the question to check that they are giving the required angle.

(d) Most correct answers came from using the formula for the distance of a point from a plane, with the common errors being with signs.

Question 6

(a) Several candidates tried to find the coordinates of the point which is furthest from either the initial line or the half-line $\theta = \frac{\pi}{2}$. Those who found $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ could form an appropriate trigonometric equation and solve it to find the relevant value of θ. Polar coordinates must be given in the form (r, θ).

(b) Many understood that the graph was a single loop in the first quadrant but only some recognised that the curve is tangential to $\theta = \frac{\pi}{2}$ at the pole.

(c) The correct formula with correct limits was applied here in most cases. The idea of using double angles to integrate squares of sin or cos was used accurately. Some candidates recognised that $\sin\theta\cos^2\theta$ can be integrated either by inspection or by using a substitution.

Question 7

(a) Most candidates could find the vertical asymptotes. The horizontal asymptote of $y = 0$ was included by the majority.

(b) This was well answered with only occasional errors in arithmetic.

(c) Almost all could find the intersections with the axes. There were many good graphs with a carefully drawn curve, with axes and asymptotes clearly labelled. Some candidates recognised the significance of the stationary points found in **part (b)** and drew a curve that approached the negative y axis from above.

(d) The concept of reflecting the graph in the x axis to find the graph of the modulus was well demonstrated and applied. The approach of the line to the asymptote always needs care.

The most successful way to deal with the inequality was to split it in two, and this was usually seen. It is often most straightforward to find the critical values of x by using equations rather than inequalities. There were 4 such values and exact answers were required, with the majority of candidates including the value $x = 0$. The set of values of x satisfying the given inequality can then be determined using the graph.

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus and there were many scripts of a high standard. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a good understanding of linear algebra. Sometimes candidates did not fully justify their answers and reached conclusions without supporting justification, particularly where answers were given within the question.

Comments on specific questions

Question 1

This question was answered very well. Most candidates accurately differentiated twice to find the Maclaurin's series, though a few worked with an erroneous value of $f'(0)$ without checking their work.

Question 2

A few candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix. Better responses maintained accuracy throughout the solution, both when manipulating the characteristic equation and when substituting $\mathbf{A}$ and $\mathbf{A}^3$.

Question 3

(a) The first part of this question was answered well by most candidates.

(b) Candidates who used the chain rule were usually successful. It was not uncommon for those who chose to multiply out the expression to introduce errors during this process. A small number of candidates made y' the subject before differentiating using the quotient rule and were usually successful also.

Question 4

(a) The more successful approaches formed a correct expression for the sum of the areas of the rectangles and integrated by parts to accurately derive the given result.

(b) Some candidates recognised that they could adapt their solution to **part (a)** and by accurately doing so generally went on to derive a suitable lower bound.

Question 5

Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) Most candidates used de Moivre's theorem successfully to find $\operatorname{cosec}5\theta$. Some candidates has difficulty translating their expression in cos and sin into an expression in cosec. There were some elegant uses of $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and good candidates showed full details of their working.

(b) Some candidates made the connection with the equation given in the first part and, justifying the relationship, used that $\operatorname{cosec}5\theta = 2$ to find all five solutions in an efficient manner.

Question 7

(a) This was done well with the majority of candidates dividing through by $\sqrt{x^2-1}$ and then arriving at the given integrating factor using the logarithmic form $\cosh^{-1}x$.

(b) Better responses maintained accuracy throughout, with candidates fully simplifying the right hand side of the equation after multiplying by the integrating factor. The integral of the right hand side could then clearly be seen as $\sqrt{x^2-1}+\mathrm{C}$ and the initial conditions substituted to find the correct value of C.

Question 8

(a) This part of the question was well done with the majority of candidates accurately substituting cosh in terms of exponentials and writing out the full expansion of $\cosh^2 A$ to justify the given identity.

(b) (i) Most candidates accurately recalled the formula for surface area, applying $\cosh^2 A = \sinh^2 A + 1$ after expanding to reach the given answer.

(ii) After multiplying out $(2\cosh 2t + 3t)\cosh 2t$, the best responses maintained accuracy by applying integration by parts and the necessary hyperbolic identities. Appling integration by parts before or after expanding were the two different approaches seen, with the latter approach being more efficient since $t\cosh 2t$ gives an odd function. A few candidates gave an answer involving sinh or cosh instead of converting their answer to one in terms of π and e.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus and there were many scripts of a high standard. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a good understanding of linear algebra. Sometimes candidates did not fully justify their answers and reached conclusions without supporting justification, particularly where answers were given within the question.

Comments on specific questions

Question 1

(a) This part was very well answered with most candidates accurately finding the Maclaurin's series using standard results from the list of formulae. Those who instead attempted to find the first four derivatives often made errors.

(b) Candidates who had found the correct series in **part (a)** usually went on to state the correct value of the fourth derivative when $x = 0$. A few incorrectly multiplied their coefficient of x^4 by 1 or 2, rather than 24.

(c) The majority of candidates integrated their answer to **part (a)** with most maintaining accuracy.

Question 2

This was done well with almost all finding the correct integrating factor. Most candidates remembered to add an arbitrary constant after integrating the right hand side, going on to substituting the initial conditions and making *y* the subject.

Question 3

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and applied the standard result for the sum of squares to accurately derive the given result.

(b) Most candidates correctly adapted their solution to **part (a)** and derived a suitable lower bound.

Question 4

(a) Almost all candidates wrote down the five fifth roots of unity.

(b) After expanding $(\cos\theta + \mathbf{i}\sin\theta)^4$ using the binomial expansion, most candidates grouped together terms contributing to the real part before applying the identity $\cos^2\theta + \sin^2\theta = 1$ to fully justify the given result. An alternative approach seen by a few candidates involved working from the right hand side to the left hand side, applying $2\cos\theta = z + z^{-1}$, and this was usually successful too.

(c) Almost all factorised correctly, linking with the previous parts of the question. Better responses included the step of solving $\cos 4\theta = 0$ and $\cos^5\theta = 1$, leading to exactly five distinct real roots.

Question 5

(a) Most candidates accurately recalled the formula for arc length with correct limits. Some candidates fully simplified $\sqrt{\dot{x}^2 + \dot{y}^2}$ before substituting into the formula, justifying the given answer and using this approach led to fewer errors when integrating.

(b) The majority of candidates found the first derivative correctly using parametric differentiation. The attempts to find the second derivative varied in length, with better responses showing the required level of algebraic fluency and remembering to divide by $\frac{dx}{dt}$ after differentiating with respect to *t*.

Question 6

(a) A few candidates spent time finding $\det(\mathbf{P} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix. Some candidates were able to maintain accuracy throughout their whole solution, both when manipulating the characteristic equation and when substituting in for $\mathbf{P}^2$.

(b) Most candidates successfully applied $\mathbf{A} = \mathbf{PDP}^{-1}$ using their answer to **part (a)**. A few candidates took the lengthy approach of finding $\mathbf{P}^{-1}$ again without using the characteristic equation.

(c) More efficient solutions included a statement that the eigenvectors do not change, with some candidates then cubing to get the new eigenvalues.

Question 7

(a) This was well done with candidates finding the first two derivatives of y or, alternatively, w with respect to x and substituting to justify the given equation.

(b) Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and some problems with notation. A few candidates gave expressions instead of equations as their answer. Better responses included the step of substituting $y = x^2 w$ and making w the subject.

Question 8

(a) This part of the question was well done with the majority of candidates accurately substituting tanh and sech in terms of exponentials and writing over a common denominator to justify the given identity.

(b) The majority of candidates used the given substitution correctly. A few did not add an arbitrary constant to their answer.

(c) Most candidates separated the integral into the correct parts, using **part (b)**. Some candidates maintained accuracy throughout, clearly applying $\tanh^2 x = 1 - \operatorname{sech}^2 x$ to derive the given reduction formula.

(d) This part was well done with most candidates accurately applying the reduction formula.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus and there were many scripts of a high standard. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed a good understanding of linear algebra. Sometimes candidates did not fully justify their answers and reached conclusions without supporting justification, particularly where answers were given within the question.

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Question 1

This question was answered very well. Most candidates accurately differentiated twice to find the Maclaurin's series, though a few worked with an erroneous value of $f'(0)$ without checking their work.

Question 2

A few candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix. Better responses maintained accuracy throughout the solution, both when manipulating the characteristic equation and when substituting $\mathbf{A}$ and $\mathbf{A}^3$.

Question 3

(a) The first part of this question was answered well by most candidates.

(b) Candidates who used the chain rule were usually successful. It was not uncommon for those who chose to multiply out the expression to introduce errors during this process. A small number of candidates made y' the subject before differentiating using the quotient rule and were usually successful also.

Question 4

(a) The more successful approaches formed a correct expression for the sum of the areas of the rectangles and integrated by parts to accurately derive the given result.

(b) Some candidates recognised that they could adapt their solution to **part (a)** and by accurately doing so generally went on to derive a suitable lower bound.

Question 5

Almost all knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

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(a) Most candidates used de Moivre's theorem successfully to find $\operatorname{cosec} 5\theta$. Some candidates has difficulty translating their expression in cos and sin into an expression in cosec. There were some elegant uses of $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and good candidates showed full details of their working.

(b) Some candidates made the connection with the equation given in the first part and, justifying the relationship, used that $\operatorname{cosec} 5\theta = 2$ to find all five solutions in an efficient manner.

Question 7

(a) This was done well with the majority of candidates dividing through by $\sqrt{x^2 - 1}$ and then arriving at the given integrating factor using the logarithmic form $\cosh^{-1} x$.

(b) Better responses maintained accuracy throughout, with candidates fully simplifying the right hand side of the equation after multiplying by the integrating factor. The integral of the right hand side could then clearly be seen as $\sqrt{x^2 - 1} + \mathrm{C}$ and the initial conditions substituted to find the correct value of C.

Question 8

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(b) (i) Most candidates accurately recalled the formula for surface area, applying $\cosh^2 A = \sinh^2 A + 1$ after expanding to reach the given answer.

(ii) After multiplying out $(2\cosh 2t + 3t)\cosh 2t$, the best responses maintained accuracy by applying integration by parts and the necessary hyperbolic identities. Appling integration by parts before or after expanding were the two different approaches seen, with the latter approach being more efficient since $t\cosh 2t$ gives an odd function. A few candidates gave an answer involving sinh or cosh instead of converting their answer to one in terms of π and e.

FURTHER MATHEMATICS

Paper 9231/31
Paper 31

Key messages

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram if preferred.

General comments

Many candidates realised the correct approach to use to answer the questions, but not all implemented it correctly, and often did so after drawing a suitable diagram.

It would help candidates if they defined the symbols they were using, in line with mathematical practice, unless the definitions are specified in the text.

For example, in **Question 6**, candidates who drew a clear diagram labelled with all forces and distances used it effectively to correctly set up Newton's equation as well the equation for the conservation of energy. As another example, the candidates who labelled the velocities in the diagram of **Question 7** almost invariably scored full marks in **part (a)**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Many candidates correctly wrote at least one equation for the tension. Most of those who wrote both equations correctly, obtained the correct answer, showing good algebraic skills.

The most common mistake was to omit to include the extension of the string x in the expression for the radius of the circle, thus writing $T = \frac{4mga}{3a}$, instead of the correct $T = \frac{4mga}{3(a+x)}$. This resulted in a linear equation in x, and not in the correct quadratic equation $9x^2 + 9ax - 4a^2 = 0$.

Question 2

Most candidates managed to separate the variable and integrate the differential equation, often including the modulus for the velocity as $\ln|v| = \ln t - t^2 + c$. However, only some candidates wrote that, in order for the argument of the natural logarithm to be a positive quantity, the solution had to be written as $\ln(-v) = \ln t - t^2 + c$, thus leading to $v = -te^{-t^2+c} = -Ate^{-t^2}$.

All the candidates who integrated the differential equation to obtain an equation of the form $v = \pm\, te^{-t^2+c} = \pm\, Ate^{-t^2}$ correctly substituted it in the given equation for the acceleration $a = \dfrac{v(1-2t^2)}{t}$, to obtain $a = \dfrac{\pm\, te^{-t^2+c}(1-2t^2)}{t} = \dfrac{\pm\, Ate^{-t^2}(1-2t^2)}{t}$, where the negative sign is correct. Many candidates selected the positive sign for this expression and then went on to use the initial condition $a(1) = 5$, thus obtaining $A = e^c = \pm 5e$. A negative expression resulted from this, having started from an incorrect formula for the velocity v. Very few candidates realised that $A = e^c$ cannot take a negative value.

A popular alternative approach was to substitute the initial condition in the given equation for the acceleration $a = \dfrac{v(1-2t^2)}{t}$, to correctly obtain $v = -5$, and then to substitute it into their solution from **part (a)**, thus obtaining $A = e^c = \pm 5e$, where, again, the positive sign is correct, although many candidates again worked with the negative sign.

Question 3

Most of the candidates identified the correct approach to answer this question, using the conservation of energy. These candidates found the gain in kinetic energy (mga) and gravitational potential energy $\left(\dfrac{1}{3}mga\right)$, as well as the initial elastic potential energy $\left(\dfrac{6mge^2}{a}\right)$. The many candidates who wrote the correct equation for the conservation of energy almost invariably obtained the correct answer showing good algebraic manipulative skills.

The most common mistake was to select an incorrect approach, for example applying Newton's second law.

Question 4

(a) This part question was answered correctly by many candidates. They correctly calculated the moments and confidently manipulated the equation to obtain the correct answer. A typical error was to express the distance of the centre of mass of triangle *BEC* from the line segment *AD* as $\left(3a - \dfrac{2}{3}h\right)$ instead of $\left(3a - \dfrac{1}{3}h\right)$. A few candidates factorised the numerator of their answer and simplified it to obtain $\bar{x} = \dfrac{9a-h}{6}$, which greatly simplified their work in **part (b)**.

Some candidates obtained the last mark in an elegant and efficient way by stating that $\bar{x} = \bar{y}$ because of the symmetry of the lamina. Other candidates did not spot this symmetry and calculated $\bar{y}$ using the same approach as for $\bar{x}$, sometimes incorrectly.

(b) Some candidates annotated the given diagram to answer this part question. Most of them identified the correct inequality $\bar{x} \leqslant 3a - h$ and then obtained the correct answer. The most common mistake in this case was to use a strict inequality.

Question 5

This question proved particularly challenging for most candidates, who struggled to express the condition that the direction of motion of the particle at point B was perpendicular to its direction of motion at point A.

Some candidates did not realise that the direction of motion was the ratio of the vertical and horizontal components of the velocity of the particle and instead worked on the displacement of the particle.

Other candidates obtained the correct answer, often in a very elegant way, for example by equating the scalar product of the velocity vectors at points A and B to zero, and then solving the resulting quadratic equation.

Question 6

(a) To obtain the correct answer for this part question, candidates were required to perform three steps: use Newton's second law at points A and B to obtain the expressions for the tension of the string at those points; use the conservation of energy to derive the equation for the speed of the particle at point B; and finally combine the two equations for the tensions using the given ratio to obtain the correct answer, having substituted the expression for the speed of B into the equation of the tension there.

Many candidates identified the steps and were able to obtain the correct answer, often in an elegant way.

The most common mistakes were to omit one gravitational potential energy term in the equation for the conservation of energy or to omit to multiply $2mga\cos\theta$ by 2 when eliminating $\frac{1}{2}$ from the equation $\frac{1}{2}m\times 5ag-\frac{1}{2}mv^2=mga\times 2\cos\theta$, thus obtaining $v^2=5ga-2ga\cos\theta$ instead of the correct $v^2=5ga-4ga\cos\theta$.

(b) This part question proved more challenging than **part (a)**. Many candidates realised that the particle had the greatest speed at the bottom of the circle, but then did not go on to use this information to create an equation involving the energy of the particle at the that point and at point A or B. Those who did usually had no difficulties in obtaining the correct answer in a variety of alternative forms.

Question 7

(a) This part question proved challenging for the candidates who did not annotate the diagram that was provided, for example by indicating the velocity after the first and the second collision. A typical error was to try to work backwards from the given answer, instead of applying the conservation of momentum and the law of restitution.

(b) The same considerations expressed in **part (a)** apply to this part question. More specifically, a small proportion of candidates stated that the final direction of the particle was parallel to the initial one, without providing any evidence to support this statement. Some of them did not annotate the diagram and did not attempt to apply the conservation of momentum and the restitution law.

Some candidates used the above-mentioned laws to show that $\tan\alpha=\frac{1}{\tan\gamma}$ and then went on to show that, after the second rebound, the direction of motion is parallel to the initial path, often in a concise and effective way.

Only a very few candidates answered this part question correctly. Their work was often laid down very clearly, expressing both components of the velocity of the particle after the second rebound in terms of the angle α, and them using them to calculate the particle's KE and the value of the coefficient of restitution e. Finally, they combined the result from **part (a)** and the fact that $\alpha+\beta=90°$ to create and successfully solve an equation in $\tan\alpha$ and e, often in a way that was both elegant and pleasing to follow.

FURTHER MATHEMATICS

Paper 9231/32
Paper 32

Key messages

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram if preferred.

General comments

In most questions the majority of candidates understood what method to use, however they omitted to draw a suitable diagram, or to annotate the given diagram, and this resulted in writing incorrect equations. This was particularly the case in **Question 5(a)** and **Question 7**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. This was particularly the case in **Questions 3(a)**, **Question 4(a)** and **Question 6(a)**.

Comments on specific questions

Question 1

(a) This part was answered correctly by the majority of candidates. Some candidates did not use the notation given in the question, using t instead of T. A few candidates gave the components of the displacement rather than the components of the velocity.

(b) Most candidates attempted this part of the question, but only a minority were able to make the correct link between the components of the velocity initially and those after time T. A common error was to omit the minus sign when inverting the fraction involving one of the velocities.

(c) This part was answered correctly by about half of the candidates who had successfully negotiated **part (b)**.

Question 2

The majority of candidates knew what they needed to do in this question and gave clear and accurate solutions. Candidates needed to consider first the collision between P and Q and apply the principle of conservation of linear momentum. This was usually done successfully. The next step was to use the energy equation, equating the gain in elastic potential energy to the loss in kinetic energy. The elastic potential energy term was usually correct, but there were often errors in the kinetic energy term. A common error was to use only the single mass m of P with the calculated velocity of the combined mass. The final step was to simplify the energy question to arrive at a quadratic equation. Algebraic errors crept in during this simplification. Candidates are reminded that they need to proceed carefully and accurately with algebra and arithmetic.

Question 3

(a) This request required candidates to equate the vertical forces at each of A and B and then combine the two expressions. As this was a given result, four lines of working were expected; two resolution equations, one step of working (usually division) and then the final answer. A significant minority of candidates wrote only $3mg\cos\theta = mg$, $\cos\theta = \frac{1}{3}$.

(b) The majority of candidates gained full marks in this part of the question. The remaining candidates knew what they needed to do but were unable to accurately use the geometry of the situation to find the radius of the horizontal circles completed by particle A. The common error was to assume that the length of AR is equal to a whereas it is the length of the entire string that is equal to a.

Question 4

This question was answered well by the majority of candidates, with many excellent, clear and concise solutions.

(a) Candidates had few difficulties in applying the method for finding the centre of mass of a solid made up of two parts to this situation. An error that occurred sometimes was to use the volume of a sphere instead of that of a hemisphere. Some candidates did not convincingly show the given result. In these cases there was an unreasonable jump from the penultimate line of working to the given result. Candidates are reminded that when an answer is given in the question, they should give sufficient working to convince the Examiner that they have achieved the given result.

(b) Most candidates understood the geometry of the situation, wrote down the expression $\tan\theta = \frac{\bar{x}}{a}$ and used the result of **part (a)** to find the two correct values of k. A few candidates had the expression for $\tan\theta$ inverted, and others made algebraic slips when simplifying.

Question 5

(a) The majority of candidates were able to write down correct equations for the conservation of momentum and Newton's experimental law along the line of centres. They then had to solve the equations to find the speed of B after the collision. A common error was to have inconsistent signs in the momentum and restitution equations. Most candidates stated the correct expression for the component of B's velocity perpendicular to the line of centres after the collision and used it to find the angle between the direction of motion of B after the collision and the line of centres. The question asks for the angle through which the direction of motion of B is deflected as a result of the collision. Many candidates made no comment on this and left their answer for the angle between the direction of motion of B and the line of centres as their final answer. In fact this angle needs to be subtracted from $120°$. Numerically this gives the same value of $60°$ but evidence that the candidate had understood this was expected for a complete solution.

(b) Most candidates approached this part by calculating the total kinetic energy of the spheres after the collision and subtracting from it the total kinetic energy before the collision. About half of the candidates obtained the correct answer. A common error was to omit the component of the velocity of sphere B after the collision. Other candidates made arithmetical slips and some used incorrect masses. It was pleasing to note that some candidates found the velocity of sphere B was equal to its velocity before the collision, and so all that was needed was the loss in the kinetic energy of sphere A.

Question 6

(a) Most candidates used Newton's second law to set up a differential equation. They then separated the variables and integrated, evaluated the constant of integration and rearranged to find v^2 in terms of x. Many candidates did this correctly. Some candidates had a sign error in the differential equation and other candidates made arithmetical slips during their solution. The final mark was for taking the square root of the expression for v^2 and justifying the minus sign in the given answer by reference to the initial conditions. A significant minority did not explain why they had taken the negative solution.

(b) Most candidates integrated the result from **part (a)** by separating the variables and found the value of the constant of integration. Many of these candidates were able to remove the logarithms successfully and hence find x in terms of t. Some candidates were unable to make any progress with the integration, not recognising its form. Others used partial fractions to facilitate the integration, though this made their solutions unnecessarily long.

Question 7

(a) This question involves a particle moving in a vertical circle that becomes detached from its string and then begins to move as a projectile. The instruction was to use the equation of the trajectory to find an expression for the speed of the particle when it begins this projectile motion. Most candidates did as the question instructed, but a significant number were not able to find correctly the coordinates of the starting point (x, y) of the projectile motion. A diagram would have been a helpful way of understanding what was happening and communicating this understanding.

(b) This part of the question required the use of an energy equation for the motion of the particle from the lowest point of its circular motion to the point where it became detached. This leads to an expression for the initial speed in terms of the velocity found in **part (a)**. Newton's second law then leads to an expression for the tension in the string. Most candidates employed the correct method to good effect. Some candidates were confused about which velocity they had found in **part (a)** and used that directly in the equation for Newton's second law, without any use of the energy equation.

FURTHER MATHEMATICS

Paper 9231/33
Paper 33

Key messages

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram if preferred.

General comments

Many candidates realised the correct approach to use to answer the questions, but not all implemented it correctly, and often did so after drawing a suitable diagram.

It would help candidates if they defined the symbols they were using, in line with mathematical practice, unless the definitions are specified in the text.

For example, in **Question 6**, candidates who drew a clear diagram labelled with all forces and distances used it effectively to correctly set up Newton's equation as well the equation for the conservation of energy. As another example, the candidates who labelled the velocities in the diagram of **Question 7** almost invariably scored full marks in **part (a)**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Many candidates correctly wrote at least one equation for the tension. Most of those who wrote both equations correctly, obtained the correct answer, showing good algebraic skills.

The most common mistake was to omit to include the extension of the string x in the expression for the radius of the circle, thus writing $T = \frac{4mga}{3a}$, instead of the correct $T = \frac{4mga}{3(a+x)}$. This resulted in a linear equation in x, and not in the correct quadratic equation $9x^2 + 9ax - 4a^2 = 0$.

Question 2

Most candidates managed to separate the variable and integrate the differential equation, often including the modulus for the velocity as $\ln|v| = \ln t - t^2 + c$. However, only some candidates wrote that, in order for the argument of the natural logarithm to be a positive quantity, the solution had to be written as $\ln(-v) = \ln t - t^2 + c$, thus leading to $v = -te^{-t^2+c} = -Ate^{-t^2}$.

All the candidates who integrated the differential equation to obtain an equation of the form $v = \pm te^{-t^2+c} = \pm Ate^{-t^2}$ correctly substituted it in the given equation for the acceleration $a = \frac{v(1-2t^2)}{t}$, to obtain $a = \frac{\pm te^{-t^2+c}(1-2t^2)}{t} = \frac{\pm Ate^{-t^2}(1-2t^2)}{t}$, where the negative sign is correct. Many candidates selected the positive sign for this expression and then went on to use the initial condition $a(1) = 5$, thus obtaining $A = e^c = \pm 5e$. A negative expression resulted from this, having started from an incorrect formula for the velocity v. Very few candidates realised that $A = e^c$ cannot take a negative value.

A popular alternative approach was to substitute the initial condition in the given equation for the acceleration $a = \frac{v(1-2t^2)}{t}$, to correctly obtain $v = -5$, and then to substitute it into their solution from **part (a)**, thus obtaining $A = e^c = \pm 5e$, where, again, the positive sign is correct, although many candidates again worked with the negative sign.

Question 3

Most of the candidates identified the correct approach to answer this question, using the conservation of energy. These candidates found the gain in kinetic energy (mga) and gravitational potential energy $\left(\frac{1}{3}mga\right)$, as well as the initial elastic potential energy $\left(\frac{6mge^2}{a}\right)$. The many candidates who wrote the correct equation for the conservation of energy almost invariably obtained the correct answer showing good algebraic manipulative skills.

The most common mistake was to select an incorrect approach, for example applying Newton's second law.

Question 4

(a) This part question was answered correctly by many candidates. They correctly calculated the moments and confidently manipulated the equation to obtain the correct answer. A typical error was to express the distance of the centre of mass of triangle *BEC* from the line segment *AD* as $\left(3a - \frac{2}{3}h\right)$ instead of $\left(3a - \frac{1}{3}h\right)$. A few candidates factorised the numerator of their answer and simplified it to obtain $\bar{x} = \frac{9a-h}{6}$, which greatly simplified their work in **part (b)**.

Some candidates obtained the last mark in an elegant and efficient way by stating that $\bar{x} = \bar{y}$ because of the symmetry of the lamina. Other candidates did not spot this symmetry and calculated $\bar{y}$ using the same approach as for $\bar{x}$, sometimes incorrectly.

(b) Some candidates annotated the given diagram to answer this part question. Most of them identified the correct inequality $\bar{x} \leqslant 3a - h$ and then obtained the correct answer. The most common mistake in this case was to use a strict inequality.

Question 5

This question proved particularly challenging for most candidates, who struggled to express the condition that the direction of motion of the particle at point B was perpendicular to its direction of motion at point A.

Some candidates did not realise that the direction of motion was the ratio of the vertical and horizontal components of the velocity of the particle and instead worked on the displacement of the particle.

Other candidates obtained the correct answer, often in a very elegant way, for example by equating the scalar product of the velocity vectors at points A and B to zero, and then solving the resulting quadratic equation.

Question 6

(a) To obtain the correct answer for this part question, candidates were required to perform three steps: use Newton's second law at points A and B to obtain the expressions for the tension of the string at those points; use the conservation of energy to derive the equation for the speed of the particle at point B; and finally combine the two equations for the tensions using the given ratio to obtain the correct answer, having substituted the expression for the speed of B into the equation of the tension there.

Many candidates identified the steps and were able to obtain the correct answer, often in an elegant way.

The most common mistakes were to omit one gravitational potential energy term in the equation for the conservation of energy or to omit to multiply $2mga\cos\theta$ by 2 when eliminating $\frac{1}{2}$ from the equation $\frac{1}{2}m\times 5ag - \frac{1}{2}mv^2 = mga\times 2\cos\theta$, thus obtaining $v^2 = 5ga - 2ga\cos\theta$ instead of the correct $v^2 = 5ga - 4ga\cos\theta$.

(b) This part question proved more challenging than **part (a)**. Many candidates realised that the particle had the greatest speed at the bottom of the circle, but then did not go on to use this information to create an equation involving the energy of the particle at the that point and at point A or B. Those who did usually had no difficulties in obtaining the correct answer in a variety of alternative forms.

Question 7

(a) This part question proved challenging for the candidates who did not annotate the diagram that was provided, for example by indicating the velocity after the first and the second collision. A typical error was to try to work backwards from the given answer, instead of applying the conservation of momentum and the law of restitution.

(b) The same considerations expressed in **part (a)** apply to this part question. More specifically, a small proportion of candidates stated that the final direction of the particle was parallel to the initial one, without providing any evidence to support this statement. Some of them did not annotate the diagram and did not attempt to apply the conservation of momentum and the restitution law.

Some candidates used the above-mentioned laws to show that $\tan\alpha = \frac{1}{\tan\gamma}$ and then went on to show that, after the second rebound, the direction of motion is parallel to the initial path, often in a concise and effective way.

Only a very few candidates answered this part question correctly. Their work was often laid down very clearly, expressing both components of the velocity of the particle after the second rebound in terms of the angle α, and them using them to calculate the particle's KE and the value of the coefficient of restitution e. Finally, they combined the result from **part (a)** and the fact that $\alpha + \beta = 90°$ to create and successfully solve an equation in $\tan\alpha$ and e, often in a way that was both elegant and pleasing to follow.

FURTHER MATHEMATICS

Paper 9231/41
Paper 41

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

The standard of responses was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 3**.

Comments on specific questions

Question 1

(a) Almost all candidates knew how to proceed with this part and many scored full marks. A few candidates used an incorrect *t*-value.

(b) Most candidates knew how to approach this part, but sign errors were very common. These occurred either from an incorrect sign in the numerator of the test statistic being considered or from an incorrectly directed inequality sign.

Question 2

(a) Most candidates differentiated accurately the given cumulative distribution function to find the probability density function. A common error was to either omit or state incorrectly the value of the probability density function outside of the range (– 1, 1).

(b) The simplest way to calculate $\mathrm{P}\left(-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}\right)$ is to recognise it as $\mathrm{F}\left(-\frac{1}{2}\right)-\mathrm{F}\left(\frac{1}{2}\right)$. A significant number of candidates chose instead to integrate the probability density function that they had found in **part (a)**. Often errors occurred during this approach.

(c) The majority of candidates introduced an extra factor x^2 into each part of the probability density function and integrated successfully. Various sign errors and algebraic errors occurred during this process.

(d) This part proved to be more challenging and less than 50 per cent of candidates obtained the correct answer. A common error in using the expression $\text{Var}(X^2) = E\left(X^4\right) - (E\left(X^2\right))^2$ was to omit the subtracted term, which is the square of the answer to **part (b)**.

Question 3

(a) Almost all candidates were able to use a binomial term to find the value of *p* correctly. Most often, the value of *q* was found by summing the expected frequencies to 150.

(b) Almost all candidates demonstrated knowledge of how to apply a goodness of fit test, but there were errors in many cases. A significant number of candidates did not combine the last four columns in the table, which was necessary because the last three expected frequencies summed to less than 5. The null hypothesis was often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypothesis. An example of an appropriate statement is 'the given normal distribution is a good model for the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

Question 4

The information given in the table represents the times before and after the course for ten athletes, and so a paired *t*-test is required. Many candidates recognised this and worked with the signed differences between the pairs of times. Common errors were an arithmetical slip in one of the subtractions and an incorrect tabular value used in the test comparison. As in other questions, the conclusion did not always include any level of uncertainty.

The question asked candidates to state any assumption that they were making in carrying out the test, but this was often omitted. Of those assumptions that were stated, many were not relevant to this question. The expected answer was that the distribution of the population differences is assumed to be normal. Unsatisfactory examples that were seen were 'it is a normal distribution', along with statements about continuity corrections, randomness and independence.

Question 5

This question was answered well by most candidates.

(a) This part was answered correctly by most candidates. Candidates were able to calculate the relevant probabilities for the number of balls that were labelled with a multiple of 3 and then use these to form a cubic polynomial for the probability generating function.

(b) This part was answered correctly by most candidates.

(c) Candidates knew that the method here was to multiply their polynomials from **parts (a)** and **(b)** to give a single polynomial of order 5. There were some algebraic errors and a few candidates added their two probability generating functions instead of multiplying them.

(d) Candidates knew that they had to differentiate their probability generating function from **part (c)** and evaluate it for $t = 1$.

Question 6

Most candidates made a good attempt at using the Wilcoxon rank-sum test. A common error was in stating the hypotheses. These need to include reference to the population median, but often the word 'population' was omitted. Sometimes, the stated hypotheses referred to the (population) mean, suggesting that although candidates knew the mechanics of the process, they were not quite sure what they were testing. Any level of uncertainty was lacking in some conclusions.

Because the sample sizes in the given information were not covered in the table of critical values for the test, a normal approximation had to be used. Most candidates were aware of this and used the correct formulae to calculate the mean and variance of the normal distribution from the sample sizes. A minority of candidates miscopied the formula for the variance, with 2 instead of 12 in the denominator. Some candidates omitted the continuity correction when finding the *z*-value to compare with the tabular value of 1.96.

FURTHER MATHEMATICS

Paper 9231/42
Paper 42

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

The standard of responses was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 2(b)** and **Question 6**.

Comments on specific questions

Question 1

Almost all candidates knew how to proceed with this question. A few candidates used the incorrect z-value of 1.645 instead of 1.96. Many candidates worked with fractions rather than the approximate decimal equivalents until the final step and thereby retained accuracy.

Question 2

(a) Most candidates knew what they had to do to show how the frequency of 13.97 was obtained. The most common error was in working to only three significant figures at some points in the calculation, often resulting in a final answer that did not round to 13.97. Some candidates included a continuity correction.

(b) Almost all candidates demonstrated knowledge of how to apply a goodness of fit test, but there were errors in many cases. A significant number of candidates did not combine the first two columns in the table, which was necessary because the first expected frequency was less than 5. The null hypothesis was often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypothesis. An example of an appropriate statement is 'the given normal distribution is a good model for the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

Question 3

The majority of candidates scored full marks on this question.

(a) Most candidates integrated the probability density function to find the value of *a*. An alternative method of finding the areas of the triangle and the trapezium under the curve of the probability density function was seen occasionally.

(b) This part was usually correct. Some arithmetical errors occurred in the integration.

(c) Almost all the candidates integrated the probability density function to find the cumulative distribution function. A common error was in omitting the constant of integration. It was necessary to include $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 2$, but often these were incorrectly combined as '$F(x) = 0$ otherwise'.

Question 4

(a) This question clearly stated that the Wilcoxon rank-sum test was required, but a significant minority of candidates attempted to use either other Wilcoxon tests or a paired-sample test. Those who followed the instruction usually obtained most of the marks. A common error was in stating the hypotheses. These need to include reference to the population median, but often the word 'population' was omitted. Sometimes, the stated hypotheses referred to the (population) mean, suggesting that although candidates knew the mechanics of the process, they were not quite sure what they were testing. The critical value of 49 was usually seen. Any degree of uncertainty was lacking in some conclusions.

(b) This part was answered well by some candidates. Few candidates recognised that a paired sample *t*-test was inappropriate because the data was not paired. The two samples in **part (a)** were not for the same people doing the two different tests. The other comment expected was that the underlying (or population) distribution was not known to be normal.

Question 5

This question was answered well by almost all the candidates.

(a) This part was answered correctly by most candidates.

(b) Again, this part was answered correctly by most candidates. The only errors were in the powers of *t*, usually with the polynomial beginning with a constant.

(c) Candidates knew that the method here was to multiply their polynomials from **parts (a)** and **(b)** to give a single polynomial of order 6. This was usually carried out accurately.

(d) Candidates knew that they had to differentiate their probability generating function of **part (c)** twice and use the relevant formula and again completed this accurately.

Question 6

This question was answered well by the majority of candidates. Solutions were usually presented clearly, with quantities evaluated at each step of the process. Candidates found the two sample variances explicitly, then the pooled variance, then the test statistic and finally compared this with the appropriate tabular value of *t*, followed by a conclusion containing a degree of uncertainty.

Some candidates chose to leave any explicit calculations until the final step of finding the test statistic. Candidates are advised to evaluate quantities as they progress through the method, retaining a good degree of accuracy as they do so. This accuracy needs to be to at least 4 significant figures if working in decimals so that their final answer is accurate to the required 3 significant figures.

Some candidates opted to use the two-sample test without assuming a pooled variance which gave the same value of the test statistic.

Credit was awarded when the assumptions were consistent with the method used. Some candidates omitted stating any assumptions. As in earlier questions, there was not always some degree of uncertainty in the conclusion.

FURTHER MATHEMATICS

Paper 9231/43
Paper 43

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

The standard of responses was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 3**.

Comments on specific questions

Question 1

(a) Almost all candidates knew how to proceed with this part and many scored full marks. A few candidates used an incorrect *t*-value.

(b) Most candidates knew how to approach this part, but sign errors were very common. These occurred either from an incorrect sign in the numerator of the test statistic being considered or from an incorrectly directed inequality sign.

Question 2

(a) Most candidates differentiated accurately the given cumulative distribution function to find the probability density function. A common error was to either omit or state incorrectly the value of the probability density function outside of the range (– 1, 1).

(b) The simplest way to calculate $\mathrm{P}\left(-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}\right)$ is to recognise it as $\mathrm{F}\left(-\frac{1}{2}\right)-\mathrm{F}\left(\frac{1}{2}\right)$. A significant number of candidates chose instead to integrate the probability density function that they had found in **part (a)**. Often errors occurred during this approach.

(c) The majority of candidates introduced an extra factor x^2 into each part of the probability density function and integrated successfully. Various sign errors and algebraic errors occurred during this process.

(d) This part proved to be more challenging and less than 50 per cent of candidates obtained the correct answer. A common error in using the expression $\text{Var}(X^2) = E(X^4) - (E(X^2))^2$ was to omit the subtracted term, which is the square of the answer to **part (b)**.

Question 3

(a) Almost all candidates were able to use a binomial term to find the value of *p* correctly. Most often, the value of *q* was found by summing the expected frequencies to 150.

(b) Almost all candidates demonstrated knowledge of how to apply a goodness of fit test, but there were errors in many cases. A significant number of candidates did not combine the last four columns in the table, which was necessary because the last three expected frequencies summed to less than 5. The null hypothesis was often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypothesis. An example of an appropriate statement is 'the given normal distribution is a good model for the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

Question 4

The information given in the table represents the times before and after the course for ten athletes, and so a paired *t*-test is required. Many candidates recognised this and worked with the signed differences between the pairs of times. Common errors were an arithmetical slip in one of the subtractions and an incorrect tabular value used in the test comparison. As in other questions, the conclusion did not always include any level of uncertainty.

The question asked candidates to state any assumption that they were making in carrying out the test, but this was often omitted. Of those assumptions that were stated, many were not relevant to this question. The expected answer was that the distribution of the population differences is assumed to be normal. Unsatisfactory examples that were seen were 'it is a normal distribution', along with statements about continuity corrections, randomness and independence.

Question 5

This question was answered well by most candidates.

(a) This part was answered correctly by most candidates. Candidates were able to calculate the relevant probabilities for the number of balls that were labelled with a multiple of 3 and then use these to form a cubic polynomial for the probability generating function.

(b) This part was answered correctly by most candidates.

(c) Candidates knew that the method here was to multiply their polynomials from **parts (a)** and **(b)** to give a single polynomial of order 5. There were some algebraic errors and a few candidates added their two probability generating functions instead of multiplying them.

(d) Candidates knew that they had to differentiate their probability generating function from **part (c)** and evaluate it for $t = 1$.

Question 6

Most candidates made a good attempt at using the Wilcoxon rank-sum test. A common error was in stating the hypotheses. These need to include reference to the population median, but often the word 'population' was omitted. Sometimes, the stated hypotheses referred to the (population) mean, suggesting that although candidates knew the mechanics of the process, they were not quite sure what they were testing. Any level of uncertainty was lacking in some conclusions.

Because the sample sizes in the given information were not covered in the table of critical values for the test, a normal approximation had to be used. Most candidates were aware of this and used the correct formulae to calculate the mean and variance of the normal distribution from the sample sizes. A minority of candidates miscopied the formula for the variance, with 2 instead of 12 in the denominator. Some candidates omitted the continuity correction when finding the *z*-value to compare with the tabular value of 1.96.